ISRAEL JOURNAL OF MATHEMATICS **123** (2001), 241–251

SYMPLECTIC SUBMANIFOLDS IN SPECIAL POSITION

BY

ROBERTO PAOLETTI*

Dipartimento di Matematica "F. Casorati", Universitá di Pavia
Via Ferrata 1, 27100 Pavia, Italy
e-mail: rp@dimat.unipv.it

ABSTRACT

By working on the symplectic blow-up, we show that the symplectic divisors produced by Donaldson in [D] may be chosen so that they contain a fixed symplectic submanifold or, in a complementary direction, so that they cut it transversally with a symplectic intersection.

1. Introduction

An **almost Hodge manifold** will mean a compact $2n$-dimensional symplectic manifold (M,ω) with cohomology class $[\omega] \in H^2(M,\mathbf{Z})/\mathrm{Tor}$. By a fundamental theorem of Donaldson, for $k \gg 0$ there is a closed symplectic submanifold $D \subseteq M$ Poincaré dual to $[k\omega]$([D], [S]). Here we shall give two results concerning how this symplectic divisor may be chosen in relation to a fixed symplectic submanifold.

To begin with, let $V \subseteq M$ be a compact $2d$-dimensional symplectic submanifold; we shall argue that if $2d < n$ Donaldson's symplectic divisor may be chosen with $D \supseteq V$, essentially by using his techniques on the blow-up $\tilde{M}$ of M along V [GS2], [MS]. Let E be the inverse image of V in $\tilde{M}$, p: $N \to V$ the normal bundle of V in M. Then N may be given a compatible complex structure so $E \cong \mathbf{P}N$.

* *Current address*: Dipartimento di Matematica "E. De Giorgi", Universitá di Lecce, Via per Arnesano, 73100 Lecce, Italy.
Received April 18, 1999

THEOREM 1.1: *Let (M, ω_M) be a closed $2n$-dimensional almost Hodge manifold and $V \subseteq M$ a compact $2d$-dimensional symplectic submanifold. Let $q: \tilde{M} \to M$ be the blow-up of V in M, $E = q^{-1}(V)$ and $\epsilon \in H^2(\tilde{M}, \mathbf{R})$ the Poincaré dual class of E. Then:*

(i) For $k \gg 0$ the cohomology class $k[q^*(\omega)] - \epsilon \in H^2(\tilde{M}, \mathbf{R})$ is represented by a symplectic form $\tilde{\omega}_k$; there exist submanifolds $\tilde{Z}_k \subseteq \tilde{M}$ symplectic for $\tilde{\omega}_k$ and Poincaré dual to it, meeting every fibre $E_x = q^{-1}(x)$, $x \in V$, in the zero locus of some holomorphic section of the hyperplane bundle.

(ii) If $2d < n$, for $k \gg 0$ there is a symplectic submanifold $Z_k \subset M$ Poincaré dual to $[k\omega_M]$ with $V \subseteq Z$.

In a complementary direction, we have:

PROPOSITION 1.1: *Let (M, ω_M) be a closed $2n$-dimensional almost Hodge manifold and $V \subseteq M$ a compact symplectic submanifold. Then for $k \gg 0$ the De Rahm cohomology class $[k\omega]$ is Poincaré dual to a symplectic submanifold $Z \subseteq M$ transversal to V and such that $V \cap Z \subseteq M$ is symplectic.*

Notation: For (X, η) a symplectic manifold, $\mathcal{J}(X, \eta)$ is the space of all almost complex structures on X compatible with η; there is a retraction $r_\eta : \mathcal{M}et(X) \to \mathcal{J}(X, \eta)$, where $\mathcal{M}et(X)$ is the space of all riemannian metrics on X. Similarly, if (E, η_E) is a symplectic vector bundle over some manifold, $\mathcal{J}(E, \eta_E)$ denotes the space of all complex structures on E compatible with η_E, and there is a retraction $r_{\eta_E}: \mathcal{M}et(E) \to \mathcal{J}(X, \eta)$ [MS].

2. Proof of Theorem 1.1.

On M there exists a hermitian line bundle H with a unitary connection ∇_H having curvature form $-2\pi i\omega_M$. Given any $J \in \mathcal{J}(M, \omega_M)$, for $k \gg 0$ Donaldson constructs a symplectic submanifold of M Poincaré dual to $[k\omega]$ as the zero locus of a section s of $H^{\otimes k}$ satisfying $|\overline{\partial}_{J,\nabla} s(x)| < |\partial_{J,\nabla} s(x)|$, $\forall x \in Z(s)$. We want to impose that $V \subseteq Z(s)$.

Let $\omega_V = \omega_M|_V$ and pick $J_V \in \mathcal{J}(V, \omega_V)$. It is well-known that there exists $J_M \in \mathcal{J}(V, \omega_M)$ such that the inclusion $V \hookrightarrow M$ is (J_V, J_M)-holomorphic [AL], [MS]. In what follows, we shall need a more explicit construction of a particular choice of J_M. Let ω_N be the symplectic structure of N. Pick $J_V \in \mathcal{J}(V, \omega_V)$ and $J_N \in \mathcal{J}(N, \omega_N)$. Then (ω_N, J_N) determine a hermitian structure h_N on N; let $P \to V$ be the $U(c)$-principal bundle of unitary frames in N, $c = n - d$, and choose a connection $\mathcal{H}(P/V) \subseteq TP$.

Let $U(c)$ act in the standard way on $\mathbf{C}^c$ and trivially on $\mathbf{C}$ and consider the projectivized action on $\mathbf{P}(\mathbf{C}^c \oplus \mathbf{C})$. Given P, the associated $\mathbf{P}^c$-bundle over V is $\wp$: $\mathcal{M} = \mathbf{P}(N \oplus \mathbf{C}) \to V$, a symplectic fibration with fibre $(\mathbf{P}^c, \Omega^{(c)})$, where $\Omega^{(c)}$ is the Fubini–Study symplectic structure of $\mathbf{P}^c$. The connection on P induces a connection $\mathcal{H}(\mathcal{M}/N) \subseteq T\mathcal{M}$. Since the action is hamiltonian, minimal coupling yields a closed compatible 2-form $\hat{\vartheta}_{\min}$ on $\mathcal{M}$ [GS1]. Then perhaps after replacing ω_M, and thus ω_V, by some sufficiently large multiple, $\omega_{\mathcal{M}} = \hat{\vartheta}_{\min} + \wp^*(\omega_V)$ is a symplectic structure on $\mathcal{M}$, for which $\mathcal{H}(\mathcal{M}/N)$ is the symplectic complement of the vertical tangent bundle $\mathcal{V}(\mathcal{M}/N)$. Thus $\omega_{\mathcal{M}} = \omega_{\mathcal{M}}^{\mathrm{hor}} \oplus \omega_{\mathcal{M}}^{\mathrm{ver}}$. There are inclusions $V \subseteq N \subseteq \mathcal{M}$, the former as the zero section and the latter as the complement of the hyperplane at infinity; the embedding $\jmath$: $V \hookrightarrow \mathcal{M}$ is symplectic and has symplectic normal bundle (N, ω_N). By the symplectic neighbourhood theorem, there are tubular neighbourhoods $V \subseteq \mathcal{S} \subseteq \mathcal{M}$, $V \subseteq S \subseteq M$ and a symplectomorphism γ: $(S, \omega_M) \to (\mathcal{S}, \omega_{\mathcal{M}})$ extending the identity on V. Setting $\mathcal{S}_\delta = (P \times B^{2c}_\delta)/U(c)$ ($B^{2c}_\delta \subseteq \mathbf{C}^c$ is the ball of radius δ centered at the origin), we may assume $\mathcal{S} = \mathcal{S}_1$.

The pair (ω_V, J_V) yields a riemannian metric g_V on V, and by pull-back a riemannian metric g'_{hor} on $\mathcal{H}(\mathcal{M}/N)$. Let us set $J_{\mathcal{M},\mathrm{hor}} = r_{\omega_{\mathcal{M},\mathrm{hor}}}(g'_{\mathrm{hor}}) \in \mathcal{J}(\mathcal{H}(\mathcal{M}/N), \omega_{\mathcal{M},\mathrm{hor}})$. Then $(\omega_{\mathcal{M},\mathrm{hor}}, J_{\mathcal{M},\mathrm{hor}})$ determine a compatible hermitian structure $h_{\mathcal{M},\mathrm{hor}}$ on $\mathcal{H}(\mathcal{M}/N)$. The vertical tangent bundle $\mathcal{V}(\mathcal{M}/N)$ carries intrinsic complex and hermitian structures compatible with $\omega_{\mathcal{M},\mathrm{vert}}$, $(J_{\mathcal{M},\mathrm{vert}}, h_{\mathcal{M},\mathrm{vert}})$, coming from the Hodge structure of $\mathbf{P}^c$. Thus, $J_{\mathcal{M}} = J_{\mathcal{M},\mathrm{hor}} \oplus J_{\mathcal{M},\mathrm{vert}} \in \mathcal{J}(\mathcal{M}, \omega_{\mathcal{M}})$ and $h_{\mathcal{M}} = h_{\mathcal{M},\mathrm{hor}} \oplus h_{\mathcal{M},\mathrm{vert}}$ is a compatible hermitian structure. Let $g_{\mathcal{M}} = \mathrm{Re}(h_{\mathcal{M}})$, a compatible riemannian structure, and let $d_{\mathcal{M}}$ be the distance function on $\mathcal{M}$ associated to $g_{\mathcal{M}}$.

Let $T \subseteq M$ be an open subset with $T \cap V = \emptyset$ and $M = S \cup T$; if $S_x = \gamma^{-1}(\mathcal{S}_x)$ $(0 \leq x \leq 1)$ and $S' = S_{9/10}$, we may take $T = M \setminus \overline{S'}$. Let $f_S + f_T = 1$ be a partition of unity on M subordinate to the open cover $\{S, T\}$. Let g_T be any riemannian metric on T and set $J_M = r_{\omega_M}(f_S\gamma^*(g_{\mathcal{M}}) + f_T g_T) \in \mathcal{J}(M, \omega_M)$. We have $J_M = \gamma^*(J_{\mathcal{M}})$ on $S_{1/2}$. In particular, the inclusion $V \hookrightarrow M$ is (J_V, J_M)-holomorphic. The pair (ω_M, J_M) determines a riemannian metric g_M, with associated distance function d_M.

Let q: $\tilde{M} \to M$ be the blow-up of M along V, constructed as in [GS2] from the data introduced above, with exceptional divisor $E = q^{-1}(V) \subseteq \tilde{M}$. Let $\tilde{S} = q^{-1}(S)$.

LEMMA 2.1: *On $\tilde{M}$ there exists a complex line bundle $\mathcal{O}_{\tilde{M}}(E)$ having a transverse section $\tilde{s}$ with zero locus E, with a hermitian structure and a unitary* *connection which is flat on the complement of $\tilde{S}$.*

Proof: The projective spaces $\mathbf{P}^c = \mathbf{P}(\mathbf{C}^c \oplus \mathbf{C})$ and $\mathbf{P}^{c-1} = \mathbf{P}(\mathbf{C}^c)$ have homogeneous coordinates $[z, \alpha]$ and $[z']$, respectively, where $z, z' \in \mathbf{C}^c$ and $\alpha \in \mathbf{C}$. Let $\tilde{\mathbf{P}}^c = \{([z,\alpha],[z'])| z \wedge z' = 0\} \subseteq \mathbf{P}^c \times \mathbf{P}^{c-1}$ be the blow-up of $\mathbf{P}^c$ at the origin, with projections π_1: $\tilde{\mathbf{P}}^c \to \mathbf{P}^c$, π_2: $\tilde{\mathbf{P}}^c \to \mathbf{P}^{c-1}$ and exceptional divisor $F = \pi_1^{-1}(0)$. Given P, we obtain a fibre bundle $\tilde{\wp}$: $\tilde{\mathcal{M}} \to V$, with fibre $\tilde{\mathbf{P}}^c$ and an induced connection $\mathcal{H}(\tilde{\mathcal{M}}/V)$, a connection preserving map of fibre bundles $\check{q}$: $\tilde{\mathcal{M}} \to \mathcal{M}$ and an exceptional divisor $\mathcal{E} = \check{q}^{-1}(V) \subseteq \tilde{\mathcal{M}}$; then $\mathcal{E} \cong \mathbf{P}(N)$. Let $\tilde{\mathcal{S}} = \check{q}^{-1}(\mathcal{S}) \subseteq \tilde{\mathcal{M}}$. If we glue $M \setminus S'$ and $\tilde{\mathcal{S}}$ along $S \setminus S'$ via γ, we obtain a manifold $\tilde{M}$, the blow-up of M along V, with a map $q : \tilde{M} \to M$ and exceptional divisor $E = q^{-1}(V) \cong \mathcal{E}$. With $\tilde{S} = q^{-1}(S)$, we have a diffeomorphism $\tilde{\gamma}$: $\tilde{S} \to \tilde{\mathcal{S}}$.

The total space of $\pi_2^* \mathcal{O}_{\mathbf{P}^{c-1}}(-1)$ consists of all triples $([z,\alpha], \ell, v)$, where $[z,\alpha] \in \mathbf{P}^c$, $\ell \in \mathbf{P}^{c-1}$, $v \in \mathbf{C}^c$ and $z \wedge \ell = v \wedge \ell = 0$; the projection onto $\tilde{\mathbf{P}}^c$ is $([z,\alpha], \ell, v) \mapsto ([z,\alpha], \ell)$. The tautological $U(c)$-invariant meromorphic section s' of $\pi_2^* \mathcal{O}_{\mathbf{P}^c}(-1)$, $s'([z,\alpha],\ell) = ([z,\alpha], \ell, z/\alpha)$, is transversal, vanishes on F and has a pole of first order along $\Lambda_\infty = (\alpha = 0)$. Thus $s = s' \otimes \alpha \in H^0(\tilde{\mathbf{P}}^c, \mathcal{O}_{\tilde{\mathbf{P}}^c}(F))$ is $U(c)$-invariant.

Now $\mathcal{O}_{\tilde{\mathbf{P}}^c}(F)$ extends to a line bundle $\mathcal{O}_{\tilde{\mathcal{M}}}(\mathcal{E})$ on $\tilde{\mathcal{M}}$; and being $U(c)$-invariant s extends to a transverse section $\check{s}$ of $\mathcal{O}_{\tilde{\mathcal{M}}}(\mathcal{E})$ vanishing on $\mathcal{E}$. Hence $\tilde{\gamma}^* \mathcal{O}_{\tilde{\mathcal{M}}}(\mathcal{E})$ may be glued to the trivial line bundle on $\tilde{M} \setminus \tilde{S}$ to obtain a line bundle $\mathcal{O}_{\tilde{M}}(E)$ on $\tilde{M}$, with a transverse section $\tilde{s}$ vanishing on E. Let $\mathcal{O}_{\tilde{M}}(-E)$ be the dual line bundle. We now produce a suitable connection on $\mathcal{O}_{\tilde{M}}(E)$.

If v lies in the fibre of $\mathcal{O}_{\tilde{\mathbf{P}}^c}(F)$ over $x \in \tilde{\mathbf{P}}^c \setminus \Lambda_\infty$, then $\frac{v}{\alpha}$ lies in the fibre of $\pi_2^* \mathcal{O}_{\mathbf{P}^{c-1}}(-1)$, hence is a vector in $\mathbf{C}^c$. Define $U(c)$-invariant hermitian metrics h_1 and h_2 for $\mathcal{O}_{\tilde{\mathbf{P}}^c}(F)$ on $\tilde{\mathbf{P}}^c \setminus \Lambda_\infty$ and $\tilde{\mathbf{P}}^c \setminus F$, respectively, by $|([z,\beta],\ell,v)|_1 = |\frac{v}{\alpha}|$ and $|s([z,\beta],\ell)|_2 = 1$. Let $\varrho_1(|z/\alpha|) + \varrho_2(|z/\alpha|) = 1$ be a partition of unity on $\mathbf{P}^c$ subordinate to the open cover $\mathbf{P}^c = B^{2c}_{2/3} \cup (\mathbf{P}^c \setminus \overline{B}^{2c}_{7/12})$, where B^{2c}_r is the ball of radius r centred at the origin of $\mathbf{C}^c \cong \mathbf{P}^c \setminus \Lambda_\infty$. Set $h = \varrho_1 h_1 + \varrho_2 h_2$, that is, $h([z,\alpha],\ell) = [\varrho_1(|z/\alpha|) \cdot |z/\alpha|^2 + \varrho_2(z)] s^* \otimes \overline{s}^*$ on $\tilde{\mathbf{P}}^c \setminus F$. Note that $\mathcal{O}_{\tilde{\mathbf{P}}^c}(F) \cong \pi_2^* \mathcal{O}_{\mathbf{P}^{c-1}}(-1)$ on $\tilde{\mathbf{P}}^c \setminus \Lambda_\infty$ and h_1 is the pull-back of the standard metric on $\mathcal{O}_{\mathbf{P}^{c-1}}(-1)$. Let ∇_h be the unique connection on $\mathcal{O}_{\tilde{\mathbf{P}}^c}(F)$ compatible with h and the holomorphic structure. Then ∇_h and its curvature form $-2\pi i \vartheta_F$ are $U(c)$-invariant. On $B^{2c}_{7/12}$, ∇_h is the pull-back of the standard connection ∇_{st} on $\mathcal{O}_{\mathbf{P}^{c-1}}(-1)$.

Arguing as in [P], given $\mathcal{H}(P/M)$ there is a connection $\nabla_{\mathcal{E}}$ on $\mathcal{O}_{\tilde{\mathcal{M}}}(\mathcal{E})$ extending ∇_h. Concretely, let θ be the connection matrix of ∇_h on $\tilde{\mathbf{P}}^c \setminus F$ in the trivialization given by s; then θ is $U(c)$-invariant, and extends to a vertical 1-form $\check{\theta}$ on $\tilde{\mathcal{M}} \setminus \mathcal{E}$, which is the connection matrix of $\nabla_{\mathcal{E}}$ with respect to $\check{s}$. Then $\tilde{\theta} = 0$ on

$\mathcal{M} \setminus \mathcal{S}$. Similarly, the curvature form $-2\pi i \vartheta_{\mathcal{E}}$ of $\nabla_{\mathcal{E}}$ extends the $U(c)$-invariant form $-2\pi i \vartheta_F$ on $\tilde{\mathbf{P}}^c$ and also vanishes on $\tilde{\mathcal{M}} \setminus \tilde{\mathcal{S}}$. Then using $\tilde{\gamma}$ we obtain a connection ∇_E on $\mathcal{O}_{\tilde{M}}(E)$, for which $\tilde{s}$ is covariantly constant on $\tilde{M} \setminus \tilde{S}$, with curvature form $-2\pi i \vartheta_E$ vanishing on $\tilde{M} \setminus \tilde{S}$. ∎

In particular, $[\vartheta_E] \in H^2(\tilde{M}, \mathbf{Z})$ is Poincaré dual to E. Let $\nabla^*_{\mathcal{E}}$ and ∇^*_E be the dual connections.

LEMMA 2.2: *The form $\omega_{\tilde{\mathcal{M}},k} = k\check{q}^*(\omega_{\mathcal{M}}) - \vartheta_{\mathcal{E}} \in \Omega^2(\tilde{\mathcal{M}})$ is symplectic for all $k \gg 0$, and $T_{\tilde{\mathcal{M}}} \cong \mathcal{H}(\tilde{\mathcal{M}}/V) \oplus \mathcal{V}(\tilde{\mathcal{M}}/V)$ is a symplectic direct sum.*

In fact, $\omega_{\tilde{\mathcal{M}},k}$ restricts vertically to $\pi_1^*(k\Omega^{(c)}) - \vartheta_F$, which is a Kähler form for $k \gg 0$. On the other hand, since $\omega_{\mathcal{M}}$ is symplectic, $\omega_{\tilde{\mathcal{M}},k}$ is symplectic on $\mathcal{H}(\tilde{\mathcal{M}}/V)$ when $k \gg 0$. Furthermore, $-2\pi i \omega_{\tilde{\mathcal{M}},k}$ is the curvature form of the product connection on $\check{q}^*(\mathcal{H}^{\otimes k}) \otimes \mathcal{O}_{\tilde{\mathcal{M}}}(-\mathcal{E})$.

COROLLARY 2.1: *$\omega_{\tilde{M},k} = kq^*(\omega_M) - \vartheta_E$ is symplectic for $k \gg 0$.*

This is so on $\tilde{S}$, since there $\omega_{\tilde{M},k} = \tilde{\gamma}^*(\omega_{\tilde{\mathcal{M}},k})$. On $\tilde{M} \setminus \tilde{S}$, on the other hand, $\omega_{\tilde{M},k} = kq^*(\omega_M)$. Clearly, $-2\pi i \omega_{\tilde{M},k}$ is the curvature form of the tensor product connection on $\tilde{H}_k = q^*(H^{\otimes k}) \otimes \mathcal{O}_{\tilde{M}}(-E)$.

Let $\mathcal{N} = \wp^*(N)$, a rank-c vector bundle on $\mathcal{M}$. Then $\mathbf{P}\mathcal{N}$ is both a $\mathbf{P}^{c-1}$-bundle over $\mathcal{M}$, t: $\mathbf{P}\mathcal{N} \to \mathcal{M}$, and a $\mathbf{P}^c \times \mathbf{P}^{c-1}$-bundle over V. In the latter interpretation, it is associated to P and the product action of $U(c)$ on $\mathbf{P}^c \times \mathbf{P}^{c-1}$, while in the former it stems from $\wp^*P$ and the action of $U(c)$ on $\mathbf{P}^{c-1}$. The inclusion $\tilde{\mathbf{P}}^c \subseteq \mathbf{P}^c \times \mathbf{P}^{c-1}$ is $U(c)$-invariant and $\mathcal{O}_{\tilde{\mathbf{P}}^c}(F) = \pi_1^*\mathcal{O}_{\mathbf{P}^c}(1) \otimes \pi_2^*\mathcal{O}_{\mathbf{P}^{c-1}}(-1)|_{\tilde{\mathbf{P}}^c}$; besides, ∇_{st} extends to a compatible connection $\nabla_{\mathcal{N}}$ on $\mathcal{O}_{\mathbf{P}\mathcal{N}}(-1)$. Therefore,

LEMMA 2.3: *We have an embedding of fibre bundles over V, ι: $\tilde{\mathcal{M}} \hookrightarrow \mathbf{P}\mathcal{N}$, and $\mathcal{O}_{\tilde{\mathcal{M}}}(\mathcal{E}) = \iota^*\Big(t^*\mathcal{O}_{\mathcal{M}}(1) \otimes \mathcal{O}_{\mathbf{P}\mathcal{N}}(-1)\Big)$. Therefore, $\check{q}^*(\mathcal{H}^{\otimes k}) \otimes \mathcal{O}_{\tilde{\mathcal{M}}}(-\mathcal{E}) \cong \check{q}^*(\mathcal{H}^{\otimes k}) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{N} \otimes \mathcal{O}_{\mathcal{M}}(1))}(1)$. The pairs $(\mathcal{O}_{\tilde{\mathcal{M}}}(\mathcal{E}), \nabla_{\mathcal{E}})$ and $(\iota^*\mathcal{O}_{\mathbf{P}\mathcal{N}}(-1), \iota^*\nabla_{\mathcal{N}})$ are isomorphic on $\mathcal{S}'' = (P \times B^{2c}_{1/2})/U(c) \subseteq \mathcal{S}'$.*

Equivalently, since $\vartheta_F = -\pi_2^*(\Omega^{(c-1)})$ on $\tilde{B}^{2c}_{7/12} = \pi_1^{-1}(B^{2c}_{7/12}) \subseteq \tilde{\mathbf{P}}^c$ and $H^1(\tilde{B}^{2n}_{7/12}, \mathbf{Q}) = 0$, there is a $U(c)$-invariant gauge equivalence on $\tilde{B}^{2c}_{7/12}$ between $(\mathcal{O}_{\tilde{\mathbf{P}}^c}(F), \nabla_h)$ and $(\pi_2^*\mathcal{O}_{\mathbf{P}^c}(-1), \pi_2^*\nabla_{\mathrm{st}})$, which globalizes to the relative situation.

To express controlled transversality, let us introduce compatible almost complex structures $J_{\tilde{\mathcal{M}},k} \in \mathcal{J}(\tilde{\mathcal{M}}, \omega_{\tilde{\mathcal{M}},k})$, $J_{\tilde{M},k} \in \mathcal{J}(\tilde{M}, \omega_{\tilde{M},k})$. Consider the decomposition $T_{\tilde{\mathcal{M}}} = \mathcal{H}(\tilde{\mathcal{M}}/V) \oplus \mathcal{V}(\tilde{\mathcal{M}}/V)$, in terms of which $\omega_{\tilde{\mathcal{M}},k} = \omega^{\mathrm{hor}}_{\tilde{\mathcal{M}},k} \oplus \omega^{\mathrm{ver}}_{\tilde{\mathcal{M}},k}$. View g_V as a riemannian metric on $\mathcal{H}(\tilde{\mathcal{M}}/V)$ and set $J^{\mathrm{hor}}_{\tilde{\mathcal{M}},k} = r_{\omega^{\mathrm{hor}}_{\tilde{\mathcal{M}},k}}(g_V) \in$

$\mathcal{J}(\mathcal{H}(\tilde{\mathcal{M}}/V), \omega^{\rm hor}_{\tilde{\mathcal{M}},k})$. With abuse of language, view the standard complex structure $J_{\tilde{\mathbf{P}}^c}$ on $\tilde{\mathbf{P}}^c$ as a complex structure on $\mathcal{V}(\tilde{\mathcal{M}}/V)$ and let

$$J_{\tilde{\mathcal{M}},k} = J^{\rm hor}_{\tilde{\mathcal{M}},k} \oplus J_{\tilde{\mathbf{P}}^c}.$$

Note that $\check{q}$: $\tilde{\mathcal{M}} \to \mathcal{M}$ is $(J_{\tilde{\mathcal{M}},k}, J_{\mathcal{M}})$-holomorphic on $\tilde{\mathcal{M}} \setminus \tilde{\mathcal{S}}$. The pair $(\omega_{\tilde{\mathcal{M}},k}, J_{\tilde{\mathcal{M}},k})$ determines a compatible riemannian metric $g_{\tilde{\mathcal{M}},k}$. On $\tilde{M}$ we may consider the auxiliary riemannian metric $g'_{\tilde{M},k} = (f_S \circ q)\tilde{\gamma}^*(g_{\tilde{\mathcal{M}},k}) + (f_T \circ q)q^*(g_T)$ and then set $J_{\tilde{M},k} = r_{\omega_{\tilde{M},k}}(g'_{\tilde{M},k})$. The pair $(\omega_{\tilde{M},k}, J_{\tilde{M},k})$ determines a compatible riemannian metric $g_{\tilde{M},k}$.

Recall the section $\check{s}$ of $\mathcal{O}_{\mathcal{M}}(\mathcal{E})$ and the connection $\nabla_{\mathcal{E}}$, with $\nabla_{\mathcal{E}}\check{s} = \check{\theta} \otimes \check{s}$. For $k \gg 0$ 1-forms on $\tilde{\mathcal{M}}$ decompose as $\Omega^1_{\tilde{\mathcal{M}}} = \Omega^{1,0}_{J_{\tilde{\mathcal{M}},k}} \oplus \Omega^{0,1}_{J_{\tilde{\mathcal{M}},k}}$, where $\Omega^{1,0}_{J_{\tilde{\mathcal{M}},k}}$ and $\Omega^{0,1}_{J_{\tilde{\mathcal{M}},k}}$ are the $\mathbf{C}$-linear and $\mathbf{C}$-antilinear forms for $J_{\tilde{\mathcal{M}},k}$, respectively. Let $\overline{\partial}_{\mathcal{E},J_{\tilde{\mathcal{M}},k}}$ be the composition of $\nabla_{\mathcal{E}}$ with the projection onto $\Omega^{0,1}_{J_{\tilde{\mathcal{M}},k}}$. By construction $\check{\theta}$ is vertical and vertically holomorphic: $\check{\theta}^{0,1}_{J_{\tilde{\mathcal{M}},k}} = 0$. Therefore, $\overline{\partial}_{\mathcal{E},J_{\tilde{\mathcal{M}},k}}\check{s} = \check{\theta}^{0,1}_{J_{\tilde{\mathcal{M}},k}} \otimes \check{s} = 0$ and $\overline{\partial}_{E,J_{\tilde{M},k}}\tilde{s} = 0$.

It is also in order to introduce auxiliary almost complex structures $J_{\tilde{\mathcal{M}}}$ and $J_{\tilde{M}}$ on $\tilde{\mathcal{M}}$ and $\tilde{M}$. These are the limits in C^1 norm of $J_{\tilde{\mathcal{M}},k}$ and $J_{\tilde{M},k}$ for $k \to \infty$. On $\tilde{\mathcal{M}}$, given the decomposition $T_{\tilde{\mathcal{M}}} = \mathcal{H}(\tilde{\mathcal{M}}/V) \oplus \mathcal{V}(\tilde{\mathcal{M}}/V)$, set $J_{\tilde{\mathcal{M}}} = J^{\rm hor}_{\mathcal{M}} \oplus J_{\tilde{\mathbf{P}}^c}$, where $J^{\rm hor}_{\mathcal{M}}$ is the horizontal component of the almost complex structure of $\mathcal{M}$, pulled-back to $\mathcal{H}(\tilde{\mathcal{M}}/V)$. Now $J^{\rm hor}_{\tilde{\mathcal{M}},k} = r_{\omega^{\rm hor}_{\tilde{\mathcal{M}},k}}(g_V)$ and $J^{\rm hor}_{\mathcal{M}} = r_{\omega^{\rm hor}_{\tilde{\mathcal{M}}}}(g_V)$; and since $||k^{-1}\omega^{\rm hor}_{\tilde{\mathcal{M}},k} - \omega^{\rm hor}_{\mathcal{M}}|| = O(1/k)$ we have $J_{\tilde{\mathcal{M}}} = J_{\tilde{\mathcal{M}},k} + O(1/k)$. Clearly $\check{q}$: $\tilde{\mathcal{M}} \to \mathcal{M}$ is $(J_{\tilde{\mathcal{M}}}, J_{\mathcal{M}})$-holomorphic. To define $J_{\tilde{M}}$, we glue J_M on $\tilde{M} \setminus \tilde{S}''$ with $\tilde{\gamma}^*(J_{\tilde{\mathcal{M}}})$. Again $J_{\tilde{M},k} = J_{\tilde{M}} + O(1/k)$ and q: $\tilde{M} \to M$ is $(J_{\tilde{M}}, J_M)$-holomorphic.

LEMMA 2.4: *There are a symplectic structure ϕ_k on $\mathbf{P}\mathcal{N}$ and $I_k \in \mathcal{J}(\mathbf{P}\mathcal{N}, \phi_k)$ such that $(\tilde{\mathcal{S}}'', \omega_{\tilde{\mathcal{M}},k}, J_{\tilde{\mathcal{M}},k})$ is a symplectic and almost complex submanifold of $(\mathbf{P}\mathcal{N}, \phi_k, I_k)$.*

Proof: The actions of $U(c)$ on $\mathbf{P}^c \times \mathbf{P}^{c-1}$ and on $\mathbf{P}^{c-1}$ are hamiltonian for, respectively, the Kähler forms $\tau_k = k\pi_1^*(\Omega^{(c)}) + \pi_2^*(\Omega^{(c-1)})$, $k > 0$, and $\Omega^{(c-1)}$. Therefore on $\mathbf{P}\mathcal{N}$ there are closed compatible minimal coupling 2-forms, $\vartheta^{(V)}_{\min,k}$ and $\vartheta^{(\mathcal{M})}_{\min}$, associated to the symplectic bundle structures over V and $\mathcal{M}$, respectively. One has $\vartheta^{(V)}_{\min,k} + k(\wp \circ t)^*(\omega_V) = \vartheta^{(\mathcal{M})}_{\min} + kt^*(\omega_{\mathcal{M}})$, a symplectic structure on $\mathbf{P}\mathcal{N}$ for $k \gg 0$, that we call ϕ_k. Then ι: $\tilde{\mathcal{M}} \subseteq \mathbf{P}\mathcal{N}$ is a symplectic submanifold for ϕ_k and $\iota^*(\phi_k)|_{S''} = \omega_{\tilde{\mathcal{M}}}|_{S''}$. Set $\phi_{\tilde{\mathcal{M}},k} = \phi_k|_{\tilde{\mathcal{M}}}$. Let us consider the direct sums $T_{\tilde{\mathcal{M}}} = \mathcal{H}(\tilde{\mathcal{M}}/V) \oplus \mathcal{V}(\tilde{\mathcal{M}}/V)$ and $T_{\mathbf{P}\mathcal{N}} = \mathcal{H}(\mathbf{P}\mathcal{N}/V) \oplus \mathcal{V}(\mathbf{P}\mathcal{N}/V)$. With abuse of language, we have riemannian metrics $G'_{\tilde{\mathcal{M}},k} = g_V \oplus g_{\tilde{\mathbf{P}}^c,k}$ and

$G'_{\mathbf{P}\mathcal{N},k} = g_V \oplus g_{\mathbf{P}^c \times \mathbf{P}^{c-1},k}$, where $g_{\mathbf{P}^c\times\mathbf{P}^{c-1},k}$ is the metric associated to τ_k, and $\iota\colon (\tilde{\mathcal{M}}, G'_{\tilde{\mathcal{M}},k}) \hookrightarrow (\mathbf{P}\mathcal{N}, G'_{\mathbf{P}\mathcal{N},k})$ is a riemannian immersion. Let us then set $S_k = r_{\iota^*\phi_k}(G'_{\tilde{\mathcal{M}},k})$ and $I_k = r_{\phi_k}(G'_{\mathbf{P}\mathcal{N},k})$. Then ι is (S_k, I_k)-holomorphic and $S_k = J_{\tilde{\mathcal{M}},k}$ on $\tilde{\mathcal{S}}''$. ∎

The compatible pair (ϕ_k, I_k) determines a riemannian structure $G_{\mathbf{P}\mathcal{N},k}$ on $\mathbf{P}\mathcal{N}$ with associated distance $d_{\mathbf{P}\mathcal{N},k}$.

Proposition 2.1: *There are $\epsilon > 0$ (independent of k) and for every $k \gg 0$ a section ρ of $\tilde{H}_k$ such that $|\bar{\partial}_{J_{\tilde{M},k}}\rho(y)|_k < C/\sqrt{k}$, $|\partial_{J_{\tilde{M},k}}\rho(y)|_k > \epsilon$ for all $y \in Z(\rho)$, where $Z(\rho) \subseteq \tilde{M}$ is the zero locus of ρ and $|\cdot|_k$ is the norm associated to $g_{\tilde{M},k}$.*

Proof: Set $d_{M,k} = \sqrt{k} d_M$ and consider, as in [D], the following real function on $M \times M$:

$$\ell_k(x,x') = \begin{cases} e^{-d_{M,k}(x,x')^2/5} & \text{if } d_{M,k}(x,x') \le k^{1/4}, \\ 0 & \text{if } d_{M,k}(x,x') > k^{1/4}. \end{cases}$$

Recall from *loc. cit.* that $\forall\ D > 0$ and $\forall k \gg 0$ there are an open cover $\mathcal{U}_k = \{U_i\}_{i=1}^s$ of M, where every U_i is the $d_{M,k}$-unit ball centred at $P_i \in M$, such that

$$(1) \qquad \sum_{i=1}^{s} d_{M,k}(P_i, q)^r \ell_k(P_i, q) \le C, \quad \forall q \in M, \quad r = 0,1,2,3;$$

and furthermore $N = N(D) \in \mathbf{N}$ independent of k and a partition $I = \{1, \ldots, s\} = \bigcup_{a=1}^N I_a$ satisfying $i, j \in I_a \Longrightarrow d_{M,k}(P_i, P_j) \ge D$. This sequence of open covers is obtained from a fixed atlas $\{\phi_l\colon O_l \to M\}$ of M, by choosing the P_i's to be, for each k, the images in M of the vertices of a suitable rescaling of the standard lattice in $\mathbf{C}^n$ [D], Lemma 12. For $k \gg 0$ and every $i = 1, \ldots, s$, let σ_i be the compactly supported, approximately J_M-holomorphic section of $(H^{\otimes k}, \nabla_{H^{\otimes k}})$ centred at P_i constructed by Donaldson ([D], Proposition 11).

Suppose $k \gg 0$ and $\mathcal{U}_k$ is an open cover of M as above. We construct an open cover $\tilde{\mathcal{U}}_k$ of $\tilde{M}$. If $P_i \notin S''$, let $\tilde{P}_i = q^{-1}(P_i) \in \tilde{M}$ and $\tilde{U}_i = q^{-1}(U_i)$. If $P_i \in S''$, there is an open neighbourhood $A \ni P_i$ on which $\gamma^*\mathbf{P}\mathcal{N}$ is trivial, hence for k large a diffeomorphism $\psi_i\colon U_i \times \mathbf{P}^{c-1} \cong \gamma^*\mathbf{P}\mathcal{N}|_{U_i}$. With loose notation, we have an embedding $\iota\colon \tilde{S} \hookrightarrow \gamma^*\mathbf{P}\mathcal{N}$ over S. Given $\ell = [v] \in \mathbf{P}^{c-1} = \mathbf{P}(\mathbf{C}^c)$, let $W_\ell = \{[w] \in \mathbf{P}^{c-1} : 2|(v,w)| \ge ||v|| \cdot ||w||\}$. We may find $\ell_1, \ldots, \ell_q \in \mathbf{P}^{c-1}$ such that $\mathbf{P}^{c-1} = \bigcup_1^q W_j$, with $W_j = W_{\ell_j}$. Let us set $\tilde{P}_{ij} = \psi_i(P_i, \ell_j)$, $j = 1, \ldots, q$, and $\tilde{U}_{ij} = \iota^{-1}(U'_{ij})$, where $U'_{ij} = \psi_i(U_i \times W_j)$. It may be that $\tilde{P}_{ij} \notin \iota(\tilde{S})$, or even that $\tilde{U}_{ij} = \emptyset$ for some i and j. Let us relabel $\tilde{U}_i$ as $\tilde{U}_{i,q+1}$, $\tilde{P}_i$ as $\tilde{P}_{i,q+1}$. Then $\tilde{\mathcal{U}}_k = \{\tilde{U}_{ij}\}$ is an open cover of $\tilde{M}$.

For $\tilde{U}_{ij} \in \tilde{\mathcal{U}}_k$, we produce a compactly supported, approximately holomorphic section of $\tilde{H}_k$. If $P_i \notin S''$, let σ_i be the compactly supported section of $H^{\otimes k}$ constructed by Donaldson; in the norm given by kg_M and with respect to $(\nabla_{H^{\otimes k}}, J_M)$, σ_i satisfies the estimates in Proposition 11 of [D]. Then $\alpha_i = q^*(\sigma_i) \otimes \tilde{s}^*$ is a compactly supported section of $\tilde{H}_k = q^*(H^{\otimes k}) \otimes \mathcal{O}_{\tilde{M}}(-E)$, and satisfies the same estimates in the norm given by $g_{\tilde{M},k}$ with respect to $(\nabla_{\tilde{H}_k}, J_{\tilde{M},k})$, where $\nabla_{\tilde{H}_k}$ is the product connection on $\tilde{H}_k$. More precisely, for $y \in \tilde{M}$

(i) for fixed R, $|\alpha_i(y)| \geq C^{-1}$ if $d_{M,k}(q(y), P_i) \leq R$;

(ii) $|\alpha_i(y)| \leq C\ell_k(q(y), P_i)$;

(iii) $|\nabla_{\tilde{H}_k}\alpha_i| \leq C[1 + d_{M,k}(q(y), P_i)]\ell_k(q(y), P_i)$;

(iv) $|\overline{\partial}\alpha_i(y)| \leq Ck^{-1/2}d_{M,k}(q(y), P_i)^2\ell_k(q(y), P_i)$;

(v) $|\nabla_{\tilde{H}_k}\overline{\partial}\alpha_i(y)| \leq Ck^{-1/2}\ell_k(q(y), P_i)\sum_{r=1}^{3} d_{M,k}(q(y), P_i)^r$.

In fact, $(k/2)q^*(g_M) < g_{\tilde{M},k} < 2kq^*(g_M)$ on $\tilde{M} \setminus \tilde{S}''$ and $\forall k \gg 0$; and furthermore $J_{\tilde{M},k}$ may be replaced with $J_{\tilde{M}}$, and thus with J_M, in estimating the relevant $\overline{\partial}$'s, with an error $O(1/k)$. Furthermore, $\tilde{s}$ is bounded below and above in $\mathcal{C}^1$-norm on $\tilde{M} \setminus \tilde{S}''$. The claimed estimates on α_i then follow directly from the corresponding ones on σ_i.

Suppose next $P_i \in S''$ and $1 \leq j \leq q$. Let $t'\colon \gamma^*\mathbf{P}\mathcal{N} \to S$ be the projection, so $t'(P_{ij}) = P_i$. By a construction in [P], for $k \gg 0$ there is a compactly supported section α_{ij} of of $t'^*(H^{\otimes k}) \otimes \gamma^*\mathcal{O}_{\mathbf{P}\mathcal{N}}(1)$, which is peaked at $\tilde{P}_{ij}$ and vertically holomorphic with respect to t' and satisfying the above estimates with respect to I_k and the product connection, in the norm $G_{\mathbf{P}\mathcal{N},k}$ [P], Lemma 2.3. On the other hand, $\gamma^*\mathcal{O}_{\mathbf{P}\mathcal{N}}(1)$ and $\mathcal{O}_{\tilde{M}}(-E)$ are gauge equivalent on $\tilde{S}''$ and $\tilde{S}''$ is a symplectic almost complex submanifold of $\mathbf{P}\mathcal{N}$; therefore α_{ij} may be interpreted as a section of $\tilde{H}_k$. Explicitly, set $x = p \circ \gamma^{-1}(P_i) \in V$ and let $x \in W_x \subseteq V$ be a neighbourhood on which N is trivial; then so are $\mathcal{N}$ and $\mathbf{P}\mathcal{N}$ on $S_x = \gamma^{-1}(p^{-1}(W_x)) \subseteq S$. Thus $\mathbf{P}\mathcal{N}|_{S_x} \cong S_x \times \mathbf{P}^{c-1}$ and $\mathcal{O}_{\mathbf{P}\mathcal{N}}|_{t^{-1}S_x} \cong \pi_2^*\mathcal{O}_{\mathbf{P}^{c-1}}(1)$. After a gauge transformation, $\alpha_{ij} = \alpha_i \otimes \varphi_j$, where $\varphi_j \in H^0(\mathbf{P}^{c-1}, \mathcal{O}_{\mathbf{P}^{c-1}}(1)) \cong (\mathbf{C}^c)^*$.

Next we provide a partition of the index set for $\tilde{\mathcal{U}}_k$,

$$\tilde{I} = \{i : p_i \notin S''\} \cup \{(i,j) : p_i \in S'', j = 1, \ldots, q\}.$$

We set $\tilde{I}_\alpha^{(j)} = \{(i,j) : p_i \in S'', i \in I_\alpha, 1 \leq j \leq q\}$ and $\tilde{I}_\alpha^{(q+1)} = \{i \in I_\alpha : p_i \notin S\}$. Then $\tilde{I} = (\bigcup_{\alpha,j} \tilde{I}_\alpha^{(j)}) \cup (\bigcup_\alpha \tilde{I}_\alpha^{(q+1)})$ is a disjoint union. By construction if $i, i' \in \tilde{I}_\alpha$, or if $(i,j), (i',j) \in \tilde{I}_{\alpha,j}$, then $d_{M,k}(p_i, p_{i'}) \geq D$. Let us order the index set $\{1, \ldots, s\} \times \{1, \ldots, q+1\}$ of the partition by $(b,i) \leq (a,j)$ if either $i \leq j$ or $i = j$ and $b \leq a$ and set $\Sigma_{(a,j)} = \bigcup_{(b,i)\leq(a,j)} U_{(b,i)}$.

If we now fix a linear combination $\rho_0 = \sum a_{ij}(0)\alpha_{ij}$, say $\rho_0 = 0$, Donaldson's iterative construction achieves controlled transversality by adjusting the coefficients in $N' = (q+1)N$ steps; at the (a,j)-th step transversality is attained for $\rho_{(a,j)}$ on $\Sigma_{(a,j)}$, that is, $|\overline{\partial}_{J_{\tilde{M},k}}\rho(y)|_k < C/\sqrt{k}$ and $|\partial_{J_{\tilde{M},k}}\rho(y)|_k > \epsilon$ for all $y \in Z(\rho) \cap \Sigma_{(a,j)}$ and fixed $C, \epsilon > 0$; here and below C and ϵ denote positive constants independent of k allowed to vary from line to line. The outcome is a section $\rho = \rho_{(s,q+1)} = \sum a_{ij}\alpha_{ij}$ of $\tilde{H}_k$ satisfying $|\overline{\partial}_{J_{\tilde{M},k}}\rho(y)|_k < C/\sqrt{k}$ and $|\partial_{J_{\tilde{M},k}}\rho(y)|_k > \epsilon$ for all $y \in Z(\rho)$, where $Z(\rho) \subseteq \tilde{M}$ is the zero locus of ρ and $|\cdot|_k$ is the norm associated to $g_{\tilde{M},k}$, as claimed. ∎

This completes the proof of statement (i) of the theorem. Suppose now that $2d < n$. Every α_{ij} is the restriction from $\gamma^*\mathbf{P}\mathcal{N}$ of a v-holomorphic section of $t'^*(H^{\otimes k}) \otimes \gamma^*\mathcal{O}_{\mathbf{P}\mathcal{N}}(1)$, and such sections are in 1-1 correspondence with smooth sections of $N_k = \gamma^*(\mathcal{N}^*) \otimes H^{\otimes k}$. Thus the span of the α_{ij}'s corresponds to a finite dimensional space W of smooth sections of N_k. Then W globally generates N_k over S'', and in particular $N^* \otimes H^{\otimes k}$ over V, since so do the α_{ij}'s for $t'^*(H^{\otimes k}) \otimes \gamma^*\mathcal{O}_{\mathbf{P}\mathcal{N}}(1)$.

With this identification, since $d < c$ some arbitrarily small perturbation of $\rho_{s,q}$ within W restricts to a nowhere vanishing section of $N^* \otimes H^{\otimes k}$, also satisfying $|\overline{\partial}_{J_{\tilde{M},k}}\rho_{s,q}(y)|_k < C/\sqrt{k}$ and $|\partial_{J_{\tilde{M},k}}\rho_{s,q}(y)|_k > \epsilon$ for all $y \in Z(\rho) \cap \Sigma_{(s,q)}$. If $S''' \subset S''$ has positive distance from $\partial S''$, for $k \gg 0$ the $\alpha_{i,q+1}$'s are supported on $M \setminus S'''$. Therefore ρ also restricts on V to a nowhere vanishing section on $N^* \otimes H^{\otimes k}$. Then the symplectic submanifold $\tilde{Z} = Z(\rho)$ of $(\tilde{M}, \omega_{\tilde{M},k})$ meets every fibre of $E \cong \mathbf{P}N \to V$ in a complex hyperplane. Thus $Z = q(\tilde{Z}) \subseteq \tilde{M}$ is a submanifold, and it remains to show that it is a symplectic Poincaré dual representative of $[k\omega_M]$.

We interpret Z as the zero locus of a smooth section of $H^{\otimes k}$. For every $\alpha_{i,q+1} = \sigma_i \otimes s$, let us set $\sigma_{i,q+1} = \sigma_i$. For $1 \leq j \leq q$, if in the appropriate gauge $\alpha_{i,j} = \alpha_i \otimes \varphi_j$, let us set $\sigma_{i,j} = \alpha_i \cdot \sigma_j$. Then Z is the zero locus of $\sigma = \sum a_{ij}\sigma_{ij}$, a section of $H^{\otimes k}$. It remains to prove that

LEMMA 2.5: *For some fixed $\epsilon > 0$ (independent of k), $|\overline{\partial}\rho(y)|_{M,k} < C/\sqrt{k}$ and $|\partial\rho(y)|_{M,k} > \epsilon$ for all $y \in Z \setminus V$.*

Proof: Let $|\cdot|_{M,k}$ and $|\cdot|_{\tilde{M},k}$ denote, respectively, the norms associated to kg_M and $g_{\tilde{M},k}$. We know that $|\overline{\partial}_{J_{\tilde{M},k}}\rho(y)|_{\tilde{M},k} < C/\sqrt{k}$ and $|\partial_{J_{\tilde{M},k}}\rho(y)|_{\tilde{M},k} > \epsilon$ for all $y \in Z(\rho)$. Since $J_{\tilde{M},k}$ approximates $J_{\tilde{M}}$ up to $O(1/k)$ in $\mathcal{C}^1$-norm, the above inequalities still hold with $J_{\tilde{M}}$ in place of $J_{\tilde{M},k}$ and, on the other hand, $J_{\tilde{M}} = q^*J_M$ on $\tilde{M} \setminus E$. Working on $M \setminus V$, we shall refer ∂ and $\overline{\partial}$ to J_M and

$J_{\tilde{M}}$. Thus, $|\overline{\partial}\rho(y)|_{\tilde{M},k} < C/\sqrt{k}$ and $|\partial\rho(y)|_{\tilde{M},k} > \epsilon$ for all $y \in Z(\rho)$.

The claimed inequality holds true on $\tilde{M} \setminus \tilde{S}'' \cong M \setminus S''$, because there $(k/2)q^*(g_M) < g_{\tilde{M},k} < 2kq^*(g_M)$ for $k \gg 0$. On S'', given the decompositions $T_{S''} = \mathcal{V}(S''/V) \oplus \mathcal{H}(S''/V)$ and $T_{\tilde{S}''} = \mathcal{V}(\tilde{S}''/V) \oplus \mathcal{H}(\tilde{S}''/V)$, every tangent or cotangent vector to $\tilde{S}'' \setminus E \cong S'' \setminus V$ splits as $v = v_{\text{hor}} + v_{\text{ver}}$. Then $|v|^2_{\tilde{M},k} = g^{\text{hor}}_{\tilde{M},k}(v_{\text{hor}}, v_{\text{hor}}) + g_{\tilde{\mathbf{P}}^c,k}(v_{\text{ver}}, v_{\text{ver}})$ and $|v|^2_{M,k} = kg^{\text{hor}}_M(v_{\text{hor}}, v_{\text{hor}}) + kg_{\mathbf{P}^c}(v_{\text{ver}}, v_{\text{ver}})$. On the other hand, $(1/2)kg^{\text{hor}}_M \leq g^{\text{hor}}_{\tilde{M},k} \leq 2kg^{\text{hor}}_M$ when $k \gg 0$, while $g_{\tilde{\mathbf{P}}^c,k} \geq kg_{\mathbf{P}^c}$ on tangent vectors implies $g_{\tilde{\mathbf{P}}^c,k} \leq kg_{\mathbf{P}^c}$ on cotangent vectors. Consider $|\overline{\partial}\rho(y)|_{M,k}$. By construction ρ is a linear combination of compactly supported sections of the form $\alpha_{ij} = \sigma_i \otimes \phi_j$ (in the appropriate gauge), with ϕ_j a linear functional on $(\mathbf{C}^c)^*$ of norm 1. As in [D], given the global control expressed by (1) it suffices to estimate each building block. Now, ϕ_j is vertically holomorphic and therefore $\overline{\partial}(\alpha_i \otimes \varphi_j) = \overline{\partial}_{\text{hor}}(\alpha_i \otimes \varphi_j) + \overline{\partial}_{\text{ver}}(\alpha_i) \otimes \varphi_j$. The first term satisfies the appropriate upper bound because the horizontal components of the metrics are equivalent and the second because so does α_i. As to $\partial\rho = \partial_{\text{hor}}\rho + \partial_{\text{ver}}\rho$, the sought lower bound on $|\partial\rho|_{M,k}$ holds for the same remark regarding the horizontal component and because $(dq)^*$ does not reduce lengths. ∎

This completes the proof of Theorem 1.1.

3. Proof of Proposition 1.1

Set $\omega_V = \omega_M|_V$ and let $J_V \in \mathcal{J}(V, \omega_V)$ and $J_M \in \mathcal{J}(M, \omega_M)$ be such that the inclusion $V \hookrightarrow M$ is (J_V, J_M)-holomorphic. Let us endow $H_V = H|_V$ with the induced connection. For $k > 0$ let $d_{V,k}$ be the distance function on V induced by the compatible pair $(J_V, k\omega_V)$, and similarly for $d_{M,k}$. Then $\frac{1}{2}d_{V,k}(x,x') \leq d_{M,k}(x,x') \leq 2d_{V,k}(x,x')$ if $x, x' \in V$ satisfy $d_{M,k}(x,x') \leq 20$, say, and $k \gg 0$.

By (a slight modification of) the arguments of [D], we may find an open cover $\mathcal{U}_k = \{U_i\}_{i=1}^s$ of M, where every U_i is the $d_{M,k}$-unit ball centred at $P_i \in M$, with the properties described in the proof of Proposition 2.1, and furthermore such that the following holds: Let $I' \subseteq I = \{1, \ldots, s\}$ be the subset of those i's for which $P_i \in V$. Then the unit balls in the metric $d_{V,k}$ centred at the P_i's with $i \in I'$ yield an open cover of V, satisfying the same type of conditions with respect to $d_{V,k}$. The partition of I' to be used is of course $I' = \bigcup_{\alpha=1}^N I'_\alpha$, where $I'_\alpha = I_\alpha \cap I'$. For $k \gg 0$ and every $i = 1, \ldots, s$, let σ_i be the compactly supported, approximately J_M-holomorphic section of $(H^{\otimes k}, \nabla_{H^{\otimes k}})$ centred at P_i constructed by Donaldson ([D], Proposition 11). Then $\sigma_i|_V$ is a compactly supported, approximately J_V-holomorphic section of $H_V^{\otimes k}$ with the restricted connection.

Given any section of the form $\sigma_0 = \sum_{i \in I} a_i \sigma_i$, where $a_i \in \mathbf{C}$ and $|a_i| \leq 1$, we may then proceed to adjust the coefficients in $2N$ steps adapting Donaldson's procedure, as follows. In the first N steps, we only modify those a_i's with $i \in I'$, so as to obtain at step N a section s_N such that $s|_V$ is η-transverse to zero on V for some $\eta > 0$. That is, at step α, $1 \leq \alpha \leq N$, we modify all the a_i's with $i \in I'_\alpha$, by applying Lemmas 18 and 19 of [D] in V. In the remaining N steps we further adjust all the coefficients, so as to ensure controlled transversality on M, but without destroying controlled transversality of the restriction to V. That is, at step $\alpha + N$, $1 \leq \alpha \leq N$, we modify all the a_i's for $i \in I_\alpha$, by applying Lemmas 18 and 19 of *loc. cit.* in M, the perturbations however being sufficiently small as to preserve controlled transversality on V. Essentially the same arguments in *loc. cit.* show that the process converges, so as to produce a section as claimed. ∎

References

[AL] M. Audin and J. Lafontaine (eds.), *Holomorphic Curves in Symplectic Geometry*, Birkhäuser, Boston, 1994.

[D] S. Donaldson, *Symplectic submanifolds and almost complex geometry*, Journal of Differential Geometry **44** (1996), 666–705.

[GS1] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, 1984.

[GS2] V. Guillemin and S. Sternberg, *Birational equivalence in the symplectic category*, Inventiones Mathematicae **97** (1989), 485–522.

[MS] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Clarendon Press, Oxford, 1995.

[P] R. Paoletti, *Symplectic subvarieties of projective fibrations over symplectic manifolds*, Annales de l'Institut Fourier **49** (1999), 1661–1672.

[S] J.-C. Sikorav, *Construction de sous-variétés sympléctiques, d'àpres S. K. Donaldson et D. Auroux*, Séminaire Bourbaki n. 844, Mars 1998.